# A note on solitary waves solutions of classical wave equations

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April 29, 2008

## Abstract

The goal of this work is to determine whole classes of solitary wave solutions general for wave equations.

## 1 Introduction

Consider a differential equation of the form:

$$\mathcal{F}(u, \frac{\partial u^r}{\partial x^r}, \frac{\partial u^s}{\partial t^s}) = 0, \tag{1}$$

The determination of travelling wave solutions of specific cases of (1), such as the Burgers or Burgers-Korteweg-De Vries equations, for instance, has been a major topic in the past few years, and play a crucial role in the study of wave equations. We presently aim at extending previous results, and give the whole classes of solitary wave solutions general for (1).

The paper is organized as follows. The general method is exposed in Section 2. A specific case is studied in section 3.

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## 2  Solitary waves

Following Feng [4] and our previous work [5], in which travelling wave solutions of the *CBKDV* equation were exhibited as combinations of bell-profile waves and kink-profile waves, we aim at determining travelling wave solutions of (1) (see [8],[9], [10], [11], [12], [13], [14], [15], [16]).

Following [4], we assume that equation (1) has travelling wave solutions of the form

$$u(x,t) = u(\xi), \quad \xi = x - v\,t \tag{2}$$

where $v$ is the wave velocity. Substituting (2) into equation (1) leads to:

$$\mathcal{F}(u, u^{(r)}, (-v)^s\, u^{(s)}) = 0, \tag{3}$$

Performing an integration of (3) with respect to $\xi$ leads to an equation of the form:

$$\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = C, \tag{4}$$

where $C$ is an arbitrary integration constant, which will be the starting point for the determination of solitary waves solutions.

In the previous works, this integration constant is usually taken equal to zero. Yet, it should not be so, since it can lead to a loss of solutions, as we are going to show it in the following.

## 3  Travelling Solitary Waves

### 3.1  Hyperbolic Ansatz

The discussion in the preceding section provides us useful information when we construct travelling solitary wave solutions for equation (3). Based on these results, in this section, a class of travelling wave solutions is searched as a combination of bell-profile waves and kink-profile waves of the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \sum_{i=1}^{n} \left( U_i \tanh^i \left[ C_i(\tilde{x} - v\,\tilde{t}) \right] + V_i \operatorname{sech}^i \left[ C_i(\tilde{x} - v\,\tilde{t} + x_0) \right] \right) + V_0 \tag{5}$$

where the $U_i's$, $V_i's$, $C_i's$, $(i = 1, \cdots, n)$, $V_0$ and $v$ are constants to be determined. In the following, $c$ is taken equal to 1.

## 3.2 Theoretical analysis

Substitution of (5) into equation (4) leads to an equation of the form

$$\sum_{i,\,j,\,k} A_i \tanh^i\left(C_i\,\xi\right) \operatorname{sech}^j\left(C_i\,\xi\right) \sinh^k\left(C_i\,\xi\right) = C \tag{6}$$

the $A_i$ being real constants.

The difficulty for solving equation (6) lies in finding the values of the constants $U_i$, $V_i$, $C_i$, $V_0$ and $v$ by solving the over-determined algebraic equations. Following [4], after balancing the higher-order derivative term and the leading nonlinear term, we deduce $n = 1$.

Then, following [5] we replace $\operatorname{sech}(C_1\,\xi)$ by $\frac{2}{e^{C_1\,\xi}+e^{-C_1\,\xi}}$, $\sinh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{2}$, $\tanh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{e^{C_1\,\xi}+e^{-C_1\,\xi}}$, and multiply both sides by $(1+e^{2\xi\,C_1})^2$, so that equation (6) can be rewritten in the following form:

$$\sum_{k=0}^{4} P_k(U_1,\ V_1,\ C_1,\ v,\ V_0)\,e^{k\,C_1\,\xi} \ = \ 0, \tag{7}$$

where the $P_k$ $(k = 0, ..., 4)$, are polynomials of $U_1$, $V_1$, $C_1$, $V_0$ and $v$.

Depending wether (6) admits or no consistent solutions, spurious solitary waves solutions may, or not, appear.

## 3.3 A specific case

Consider the specific case when (1) is the equivalent equation of a *DRP* scheme, the coefficients of which will be denoted by $\gamma_k$, $k \in \{-m, m\}$ (see [1]):

$$- u_t - \frac{\sigma}{2}\,u_{tt} + \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k\,u_x = 0 \tag{8}$$

where $Re_h$ denotes the mesh Reynolds number, $\sigma$, the *cfl* coefficient, and $\mu$, the viscosity.

Equation (3) is then given by:

$$- v\,\tilde{u}'(\xi) - \frac{v^2\,\sigma}{2}\,\tilde{u}''(\xi) + \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k\,\tilde{u}'(\xi) = 0 \tag{9}$$

Performing an integration of (9) with respect to $\xi$ yields:

$$- v\,\tilde{u}(\xi) - \frac{v^2\,\sigma}{2}\,\tilde{u}'(\xi) + \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k\,\tilde{u}(\xi) = C \tag{10}$$

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i. e.:

$$\left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \tilde{u}(\xi) - \frac{v^2\,\sigma}{2}\,\tilde{u}'(\xi) = C \tag{11}$$

where $C$ is an arbitrary integration constant.
Substitution of (5) for $n = 1$ into equation (11) leads to:

$$\left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \{U_1 \tanh\left[C_1\,\xi\right] + V_1 \operatorname{sech}\left[C_1\,\xi\right] + V_0\} - \frac{v^2\,\sigma}{2} \left\{ U_1 \operatorname{sech}^2\left[C_1\,\xi\right] - V_1 \frac{\sinh\left[C_1\,\xi\right]}{\cosh^2\left[C_1\,\xi\right]} \right\} = C \tag{12}$$

i. e.:

$$\left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \left\{ U_1 \frac{e^{C_1\,\xi} - e^{-C_1\,\xi}}{e^{C_1\,\xi} + e^{-C_1\,\xi}} + \frac{2\,V_1}{e^{C_1\,\xi} + e^{-C_1\,\xi}} + V_0 \right\} - \frac{v^2\,\sigma}{2} \left\{ U_1 \left( \frac{2}{e^{C_1\,\xi} + e^{-C_1\,\xi}} \right)^2 - 2\,V_1 \frac{e^{C_1\,\xi} - e^{-C_1\,\xi}}{\left(e^{C_1\,\xi} + e^{-C_1\,\xi}\right)^2} \right\} = C \tag{13}$$

Multiplying both sides by $\left(1 + e^{2\,C_1\,\xi}\right)^2$ yields:

$$\begin{aligned} & \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \left\{ U_1 \left(e^{4\,C_1\,\xi} - 1\right) + 2\,V_1 \left(e^{3\,C_1\,\xi} + e^{C_1\,\xi}\right) + V_0 \left(1 + e^{2\,C_1\,\xi}\right)^2 \right\} \\ & - \frac{v^2\,C_1\,\sigma\,C_1}{2} \left\{ 4\,U_1 - 2\,V_1 \left(e^{3\,C_1\,\xi} - 1\right) \right\} = C \end{aligned} \tag{14}$$

which is a fourth-order equation in $e^{C_1\,\xi}$. This equation being satisfied for any real value of $\xi$, one therefore deduces that the coefficients of $e^{k\,C_1\,\xi}$, $k = 0, \ldots, 4$ must be equal to zero, i.e.:

$$\begin{cases} 2 \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \{-U_1 + V_0\} - \frac{v^2\,C_1\,\sigma}{2} \{4\,U_1 + 2\,V_1\} = C \\[2mm] \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} 2\,V_1 = 0 \\[2mm] 2 \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} V_0 = 0 \\[2mm] 2 \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} V_1 + v^2\,C_1\,\sigma\,V_1 = 0 \\[2mm] \left\{ \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k - v \right\} \{U_1 + V_0\} = 0 \end{cases} \tag{15}$$

$v = \frac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k$, $V_1 \neq 0$ leads to the trivial null solution. Therefore, $V_1$ is necessarily equal to zero, which implies:

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$$\begin{cases} v = \dfrac{2\,\sigma}{\mu\,Re_h} \sum_{k=1}^{m} k\,\gamma_k \\[2ex] U_1 = -\dfrac{C}{2\,C_1\,v^2\,\sigma} \\[2ex] V_0 \in \mathbb{R} \ , \ \ C_1 \in \mathbb{R} \end{cases} \tag{16}$$

It is easy to note that, if the integration constant $C$ had been taken equal to zero, the solitary waves of the considered equation would have been loss.

# 4    Conclusions

The importance of choosing an integration constant which is not equal to zero, in the determination of solitary wave solutions of wave equations, has been carried out. We show that taking this constant equal to zero leads to a loss of solutions.

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